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Paths in universes having closed time-like lines

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Abstract. The equations of motion of charged particles in a few cosmological solutions are investigated. The solutions concerned have electromagnetic fields and closed time-like lines and the cosmic matter is also electrically charged. It is found that, unlike the Godal universe, in these solutions particles may under some circumstances describe closed time-like lines.

1. Introduction

In Godel's (1949) universe there exist closed time-like lines. However, Chandrasekhar and Wright (1961) showed that these closed time-like lines are not geodesics, and hence no test particle would ordinarily describe them.

Recently Ozsvath (1967) and Som and Raychaudhury (1968) have given some new line elements which also exhibit rigid rotation of the cosmic matter. In these 'universes' closed time-like lines also exist, and as there is a non-vanishing electromagnetic field the path of any test particle would depend on its charge-to-mass ratio. We investigate here the equations of motion in these universes and the principal motivation is to find out whether particles, either charged or uncharged, can actually describe closed time-like lines.

2. Equations of motion and their integration

The line element due to Ozsvath (1967) which we shall consider is

$$ds^2 = \frac{1}{\alpha_2}\left\{(dx^0)^2-(dx^1)^2-(dx^3)^2+\frac{\exp(2x^1)}{2(1-l^2)}(dx^2)^2+2\exp(x^1)\,dx^0\,dx^2\right\} \tag{1}$$

where $\alpha^2 = a^2(1-2l^2)$ and $l^2 < \frac{1}{2}$. This line element may be considered to be a generalization of that of Godal, as for $l = 0$ it passes over to the Godel form. In the Ozsvath solution there is an electromagnetic field, whose non-vanishing components are

$$F^{01} = -a^3l\{(1-l^2)(1-2l^2)\}^{1/2}, \qquad F^{21} = a^3l\{(1-l^2)(1-2l^2)\}^{1/2}\exp(-x^1)$$

and

$$F_{21} = \frac{1}{2}\frac{l}{a}\frac{\exp(x^1)}{\{(1-2l^2)(1-l^2)\}^{1/2}}.$$

The matter density ρ, pressure p and charge density σ of the fluid are respectively given by

$$\rho = \frac{a^2}{16\pi}, \qquad p = \frac{a^2(1-2l^2)}{16\pi} \qquad \text{and} \qquad \sigma = -\frac{a^2l\{(1-l^2)\}^{1/2}}{4\pi}.$$

The equation of motion of a particle can be written as

$$u^\mu_{;\nu}u^\nu = -\frac{e}{m}F^{\mu}_{\cdot\nu}u^\nu \tag{2}$$

where the charge-to-mass ratio is e/m and $u^\mu = dx^\mu/ds$. So that, writing out explicitly, we obtain

$$\frac{du^0}{ds}+\frac{\exp(x^1)}{1-2l^2}u^1u^2+\frac{2(1-l^2)}{1-2l^2}u^0u^1 = -\frac{e}{m}al\left(\frac{1-l^2}{1-2l^2}\right)^{1/2}u^1 \tag{3}$$

$$\frac{du^2}{ds}-\frac{2l^2}{1-2l^2}u^1u^2-\frac{2(1-l^2)}{1-2l^2}\exp(-x^1)u^0u^1 = \frac{e}{m}al\left(\frac{1-l^2}{1-2l^2}\right)^{1/2}u^1\exp(-x^1) \tag{4}$$

$$\frac{du^3}{ds} = 0 \tag{5}$$

$$\frac{du^1}{ds} + \frac{\exp(2x^1)}{2(1-l^2)}(u^2)^2 + \exp(x^1)u^0u^2 = -\frac{e}{m}\frac{al}{2}\left(\frac{1-2l^2}{1-l^2}\right)^{1/2} u^2\exp(x^1). \tag{6}$$

A first integral of these equations is given by the line element itself, i.e.

$$\frac{1}{\alpha^2}\left\{(u^0)^2-(u^1)^2-(u^3)^2+\frac{\exp(2x^1)}{2(1-l^2)}(u^2)^2+2\exp(x^1)u^0u^2\right\} = 1 \tag{7}$$

where we have assumed the paths to be time-like. Integration of these equations yields

$$u^0 + \exp(x^1)u^2 = \frac{C}{\sqrt{2}} \tag{8}$$

$$u^3 = C_3. \tag{9}$$

Then (7) may be written as

$$\frac{\exp(2x^1)}{2(1-l^2)}(1-2l^2)(u^2)^2+(u^1)^2 = \frac{C^2}{2}-C_3{}^2-\alpha^2 = B^2 \tag{10}$$

say, where B, C, C_3 are constants of integration and

$$\frac{C}{\sqrt{2}} > B. \tag{11}$$

Also putting

$$u^1 = B\sin\theta \tag{12}$$

we have from equations (6), (8) and (10)

$$ds = -\frac{d\theta}{C\{(1-l^2)/(1-2l^2)\}^{1/2}+(e/m)(al/\sqrt{2})-B\cos\theta}. \tag{13}$$

If we write

$$A = C\left(\frac{1-l^2}{1-2l^2}\right)^{1/2}+\frac{e}{m}\frac{al}{\sqrt{2}}$$

the inequality (11) gives $A > B$ if $el/m \geqslant 0$. Actually, both e, l and hence the product el may be of either sign and four different cases arise in the integration of (13).

Case (i): $A > B$. This case includes in particular the Godel universe $l = 0$. Equation (12) on integration yields

$$\exp(x^1) = \exp(C_1)\frac{1+\beta\tan^2\Omega}{1+\tan^2\Omega} \tag{14}$$

where $\tan\frac{1}{2}\theta = \sqrt{\beta}\tan\Omega$, $\beta = (A-B)/(A+B)$ and C_1 is a constant. A little calculation now gives

$$x^2 = \frac{2B}{(A^2-B^2)^{1/2}}\left\{\frac{2(1-l^2)}{1-2l^2}\right\}^{1/2}\exp(-C_1)\frac{\tan\Omega}{1+\beta\tan^2\Omega}+C_2 \tag{15}$$

$$x^0 = -\left(\frac{2A^2}{A^2-B^2}\right)^{1/2}\frac{\Omega}{\{(1-l^2)(1-2l^2)\}^{1/2}}+2\sqrt{2}\left(\frac{1-l^2}{1-2l^2}\right)^{1/2}\tan^{-1}(\sqrt{\beta}\tan\Omega)$$
$$-\frac{e}{m}\frac{al}{\sqrt{2}}\left(\frac{1-2l^2}{1-l^2}\right)^{1/2}\left(\frac{2}{A^2-B^2}\right)^{1/2}\Omega+C_0 \tag{16}$$

and

$$x^3 = \frac{2C_3}{(A^2-B^2)^{1/2}}\Omega+C_4. \tag{17}$$

The above equations are formally quite similar to the corresponding equations of the Godel universe obtained by Chandrasekhar and Wright (1961), and it is indeed possible to transform the line element (1) also into a cylindrically symmetric form. Then the line element (1) goes over to the form

$$ds^2 = \frac{4}{\alpha^2}[(dt)^2-(dr)^2-(dz)^2+(d\phi)^2\{(K-1)\sinh^4 r-\sinh^2 r\}+2\sqrt{K}\sinh^2 r\, d\phi\, dt] \quad (18)$$

where $K = 2(1-l^2)/(1-2l^2)$, $K \geqslant 2$. The transformation formulae are

$$\exp(x^1) = \cosh 2r+\sinh 2r\cos\phi \quad (19a)$$

$$\frac{x^2\exp(x^1)}{\sqrt{K}} = \sinh 2r\sin\phi \quad (19b)$$

$$\tfrac{1}{2}\phi+\frac{1}{2\sqrt{K}}(x_0-2t) = \tan^{-1}(e^{-2r}\tan\tfrac{1}{2}\phi) \quad (19c)$$

and

$$x^3 = 2z. \quad (19d)$$

As in Chandrasekhar and Wright's paper, it turns out that constant r lines can be obtained for $\exp(+2C_1) = 1/\beta$. In this case $e^{-2r} = \sqrt{\beta}$ and $\phi = 2\Omega$. From (19c) and (16)

$$t = \sqrt{K}\left\{1-\frac{\beta+1}{(4\beta)^{1/2}}\frac{1}{2(1-l^2)}-\frac{e}{m}\frac{al}{4}\frac{1-2l^2}{1-l^2}\left(\frac{2}{A^2-B^2}\right)^{1/2}\right\}\Omega \quad (20)$$

and for uncharged particles

$$t = \sqrt{K}\left\{1-\frac{\beta+1}{(4\beta)^{1/2}}\frac{1}{2(1-l^2)}\right\}\Omega.$$

As $\beta = (A-B)/(A+B)$, if the two limiting values of B (i.e. zero and infinity) are considered, we find that the limits of β lie between 1 and 0·172 for finite values of e/m. Then, following Chandrasekhar and Wright (1961), we can say that for the range of r allowed, the constant of proportionality between t and Ω $(= \tfrac{1}{2}\phi)$ is always positive for uncharged particles.

However, for the charged particles the constant of proportionality may vary from positive to negative values, depending on whether el/m is less or greater than

$$\left\{\frac{(4\beta)^{1/2}}{\beta+1}-\frac{1}{2(1-l^2)}\right\}A\sqrt{2}\left(\frac{K}{a}\right)$$

over the range of r allowed, and hence for suitable values of el/m, one can have a particle going 'backward' in time as envisaged in Godel's paper.

Case (*ii*): $|A| < B$. In this case el/m is always negative.
If

$$\tan\Omega = \frac{(B+A)^{1/2}\tan\tfrac{1}{2}\theta-(B-A)^{1/2}}{(B+A)^{1/2}\tan\tfrac{1}{2}\theta+(B-A)^{1/2}}$$

(13) may be written

$$ds = -\frac{1}{(B^2-A^2)^{1/2}}\frac{\sec^2\Omega}{\tan\Omega}d\Omega.$$

Then, proceeding exactly as in case (i), we obtain

$$\exp(x^1) = \exp(C_1)\left\{B\left(\tan\Omega+\frac{1}{\tan\Omega}\right)-2A\right\} \quad (21)$$

$$x^2 = \frac{A}{B^2}\frac{1}{(1-A^2/B^2)^{1/2}}\left\{\frac{2(1-l^2)}{1-2l^2}\right\}^{1/2}\exp(-C_1)\frac{\tan\Omega-B/A}{\sec^2\Omega-(2A/B)\tan\Omega}+C_2 \quad (22)$$

$$x^0 = \frac{\ln \tan \Omega}{(B^2-A^2)^{1/2}} \left\{\frac{1-2l^2}{2(1-l^2)}\right\}^{1/2} \left(\frac{A}{1-2l^2} + \frac{e}{m}\frac{al}{\sqrt{2}}\right)$$
$$-2\left\{\frac{2(1-l^2)}{1-2l^2}\right\}^{1/2} \tan^{-1}\left\{\frac{\tan\Omega - A/B}{(1-A^2/B^2)^{1/2}}\right\} + C_0. \tag{23}$$

The expression for x^3 remains the same as (17).

C_1, C_2 and C_3 are again the arbitrary constants of integration. Again, if we consider the cylindrically symmetric form of the line element, we have from (19*a*) and (19*b*)

$$\cosh 2r = \frac{1+\exp(2x^1)+(1/K)(x^2)^2\exp(2x^1)}{2\exp(x^1)}.$$

Since C_2 can be eliminated by a translation along x^2, from (21) and (22)

$$\cosh 2r = \frac{1+\exp(2C_1)\{B(\tan\Omega+1/\tan\Omega)-2A\}^2+\{A^2/(B^2-A^2)\}\{(\tan\Omega-B/A)/\tan\Omega\}^2}{2\exp(C_1)\{B(\tan\Omega+1/\tan\Omega)-2A\}}.$$

It is easy to see that one cannot have a constant r by any adjustment of the constants C^1, B and A.

Case (iii): $|A| > B$, but A is negative.

In this case, el/m is sufficiently negative. If

$$\beta = \frac{|A|+B}{|A|-B} \quad \text{and} \quad \tan\tfrac{1}{2}\theta = \sqrt{\beta}\tan\Omega$$

then, proceeding exactly as in case (i), we find the expressions for the coordinate

$$\exp(x^1) = \exp(C_1)\frac{1+\beta\tan^2\Omega}{1+\tan^2\Omega} \tag{24}$$

$$x^2 = \frac{2B\sqrt{K}}{(A^2-B^2)^{1/2}}\exp(-C_1)\frac{\tan\Omega}{1+\beta\tan^2\Omega}+C_2 \tag{25}$$

$$x^0 = \frac{2|A|}{(A^2-B^2)^{1/2}}\frac{\Omega}{\{(1-l^2)(1-2l^2)\}^{1/2}}$$
$$-\frac{e}{m}\frac{al}{\sqrt{K}}\frac{2}{(A^2-B^2)^{1/2}}\Omega - 2\sqrt{K}\tan^{-1}(\sqrt{\beta}\tan\Omega)+C_0 \tag{26}$$

and the expression for x^3 remains the same as (17). C_1, C_2 and C_0 are again the arbitrary constants of integration. Equations (24) and (25) are identical with equations (14) and (15), except that A is replaced by $|A|$, and in this case too we may have constant r lines as paths as in case (i). But (26) is different from (16) of case (i). Thus the relation between t and Ω is now quite different:

$$t = \sqrt{K}\left\{1+\frac{|A|}{(A^2-B^2)^{1/2}}\frac{1}{2(1-l^2)} - \frac{e}{m}\frac{al}{\sqrt{2K}}\frac{1}{(A^2-B^2)^{1/2}} - 2\tan^{-1}(\beta\tan\Omega)\right\}\Omega.$$

One can easily see that, as $\Omega\ (=\tfrac{1}{2}\phi)$ changes, under certain conditions $dt/d\Omega$ may pass through zero. Hence a particle may move 'backward' in time.

Case (iv): $|A| = B$ and A may be either positive or negative.

This case is not of much importance, as one can easily see that in this case constant r lines cannot be achieved. An investigation as to whether a particle can describe a time-like line (i.e. $dt/ds = dr/ds = dz/ds = 0$; $ds^2 > 0$) seems interesting. Although one can obtain the condition from (20), a direct investigation from the cylindrically symmetric form of the line element seems easier.

From equation (2) one obtains

$$\left(\frac{d\phi}{ds}\right)^2\left\{2(K-1)\sinh^2 r-1\right\} = -\frac{e}{m}\frac{al}{\sqrt{2}} \tag{27}$$

and from the line element (18)

$$\frac{4}{\alpha^2}\left(\frac{d\phi}{ds}\right)^2[\sinh^2 r\{(K-1)\sinh^2 r-1\}] = 1. \tag{28}$$

For the relations (27) and (28) to be consistent the condition $l^2(e/m)^2 \geqslant 2$ must hold good. Hence time-like ϕ lines are available only for particles with charge-to-mass ratio satisfying the condition $|e/m|^2 \geqslant 2/l^2$. Since $0 \leqslant l^2 < \frac{1}{2}$, $|e/m|$ must be greater than 2.

However, for the fluid constituting the cosmic background

$$\frac{\sigma}{\rho} = -4l(1-l^2)^{1/2}, \qquad \text{i.e.} \qquad \left(\frac{\sigma}{\rho}\right)^2 < \frac{2}{l^2} \qquad \text{as} \qquad l^2 < \tfrac{1}{2}.$$

Hence the cosmic particles cannot traverse the closed time-like ϕ lines.

We consider also the line element due to Som and Raychaudhury (1968)

$$ds^2 = dt^2 - e^{2\psi}\{(dr)^2+(dz)^2\} - l(d\phi)^2 + 2m\, d\phi\, dt$$

where

$$m = ar^2$$

$$l = r^2(1-a^2r^2)$$

and

$$\psi = \tfrac{1}{2}r^2(A^2-a^2).$$

So that in this case also there is a rigid rotation. The non-vanishing field components are

$$F^{31} = \frac{A}{r}e^{-2\psi} \qquad \text{and} \qquad F^{10} = \frac{m}{r}A\,e^{-2\psi}$$

where the t, r, z and ϕ components are designated by the suffixes 0, 1, 2 and 3 respectively.

In this case an integration of the equations of motion in general is too complicated and we investigate simply the question as to whether a particle can describe a time-like ϕ line

$$\left(\frac{dr}{ds} = \frac{dz}{ds} = \frac{dt}{ds} = 0, \qquad ds^2 > 0\right).$$

From (2) one can obtain

$$\frac{d\phi}{ds} = \frac{e}{m}\frac{A}{1-2a^2r^2}$$

and from the line element, itself,

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{1}{r^2(a^2r^2-1)}.$$

Hence

$$\left(\frac{e}{m}\right)^2 = \frac{(1-2a^2r^2)^2}{a^2r^2-1}\frac{1}{A^2r^2}.$$

It follows from the above equation that for a particle to describe a time-like ϕ line $|e/m| < |2a/A|$. However, from Som and Raychaudhury (1968) the ratio of the charge to the mass density of the fluid constituting the cosmic background is

$$\frac{\sigma}{\rho} = \frac{4Aa}{4a^2 - 2A^2}.$$

Hence the cosmic particles can describe the closed time-like ϕ line only if $a^2 > A^2$.

3. Concluding remarks

While the existence of a closed time-like line is generally taken as violating causality, it has sometimes been held that these lines are of no great significance in the Godel universe as they are not the 'natural' paths of any particle. Our investigation, however, shows that the Einstein–Maxwell equations do allow solutions where such closed time-like lines can be the paths of some particles and thus a general relativistic situation violating causality can be formulated.

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